

RESTRICTED AVERAGING OPERATORS TO CONES OVER FINITE FIELDS

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ABSTRACT. We investigate the sharp $L^p \rightarrow L^r$ estimates for the restricted averaging operator A_C over the cone C of the d -dimensional vector space \mathbb{F}_q^d over the finite field \mathbb{F}_q with q elements. The restricted averaging operator A_C for the cone C is defined by the relation that $A_C f = f * \sigma|_C$, where σ denotes the normalized surface measure on the cone C , and f is a complex valued function on the space \mathbb{F}_q^d with the normalized counting measure dx . In the previous work [15], the sharp boundedness of A_C was obtained in odd dimensions $d \geq 3$ but partial results were only given in even dimensions $d \geq 4$. In this paper we prove the optimal estimates in even dimensions $d \geq 6$ in the case when the cone $C \subset \mathbb{F}_q^d$ contains a $d/2$ dimensional subspace.

1. INTRODUCTION

Let T be an operator on the class of Schwarz functions $f : \mathbb{R}^d \rightarrow \mathbb{C}$. The main question in harmonic analysis is to determine the exponents $1 \leq p, r \leq \infty$ such that the following inequality holds:

$$(1.1) \quad \|Tf\|_{L^r(\mathbb{R}^d)} \leq C \|f\|_{L^p(\mathbb{R}^d)}$$

where the constant $C > 0$ is independent of the Schwarz functions f . For example, when Tf is the Fourier transform of f , the Hausdorff-Young inequality states that the inequality (1.1) holds for $1 \leq p \leq 2$ and $1/p + 1/r = 1$.

Another interesting question is to decide whether Tf can be meaningfully restricted to a surface $V \subset \mathbb{R}^d$ or not. More precisely, we are interested in finding exponents $1 \leq p, r \leq \infty$ such that the following restriction estimate holds:

$$\|Tf\|_{L^r(V, d\nu)} \leq C \|f\|_{L^p(\mathbb{R}^d)}$$

where $d\nu$ denotes a surface measure on $V \subset \mathbb{R}^d$. Clearly, the answer to this question relies on the surface V and the operator T . To indicate that Tf is a function restricted to the surface V , we write $T_V f$ for Tf . This problem is referred to as the restriction problem for the surface V . When $T_V f = \hat{f}$, it is well known as the Fourier restriction problem for V which was initially introduced by E.M. Stein in 1967. In particular, many researchers have made much effort on solving the conjecture on the Fourier restriction problem for the sphere, the paraboloid, and the cone. The complete answers are known for the parabola and the circle in two dimensions, and for

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the cones in three and four dimensions (see [27, 1, 25]). However, the conjecture is still open in higher dimensions and some new ideas are needed to completely understand the Fourier restriction phenomena. For the background and recent progress on the Fourier restriction problem, we refer readers to [22, 26, 3, 7, 23, 2, 21, 9, 10].

It has been believed that new approaches are needed to obtain further improvement on the Fourier restriction problem. It will be helpful to see the matter in a different point of view. Based on this mind, Mockenhaupt and Tao [20] initially studied the Fourier restriction problem in the finite field setting. Their work has been improved by other researchers (see [11, 17, 18, 19]). Other interesting problems in harmonic analysis have been formulated and studied in the finite field setting (for example, see [4, 5, 6]).

It is also important to grasp the fundamental phenomena which appear in restricting operators to an appropriate surface. One may study the restriction problem related to certain operators which are different from the Fourier transformation. The authors in [8] provided some size information about convolution functions restricted to any affine subspace in \mathbb{R}^d . In the finite field setting, the author in [13] initially investigated and obtained the sharp $L^p \rightarrow L^r$ mapping properties for the restricted averaging operator to any algebraic curve in two dimensional vector spaces over finite fields. This result was deduced by applying the sharp Fourier restriction theorem on curves in two dimensions. This work was extended to higher dimensional algebraic varieties such as the paraboloid, the sphere, and the cone. Indeed, using more delicate Fourier decay estimate, the authors in [15] established the optimal $L^p \rightarrow L^r$ estimates of the restricted averaging operator over regular varieties such as the paraboloid and the sphere in all dimensions, and the cone in odd dimensions. In addition, they obtained certain weak-type estimates for the cone in even dimensions. In this paper, we shall establish the sharp strong-type estimates for the cone in even dimensions $d \geq 6$ in the specific case when the cone C contains a $d/2$ -dimensional subspace.

1.1. Review of the discrete Fourier analysis. After reviewing the definition of the restricted averaging problem for the cone in the finite field setting, our main result will be clearly stated. Before we proceed with this, let us introduce some notation and basic concepts of the discrete Fourier analysis. Let \mathbb{F}_q^d be the d -dimensional vector space over a finite field \mathbb{F}_q with q elements. We shall always assume that q is a power of odd prime. We endow the space \mathbb{F}_q^d with the normalized counting measure dx . We write (\mathbb{F}_q^d, dx) for the vector space \mathbb{F}_q^d with the normalized counting measure. On the other hand, the dual space of (\mathbb{F}_q^d, dx) will be denoted by (\mathbb{F}_q^d, dm) which is equipped with the counting measure dm . Thus, if $f : (\mathbb{F}_q^d, dx) \rightarrow \mathbb{C}$, and $g : (\mathbb{F}_q^d, dm) \rightarrow \mathbb{C}$, then the notation of norms is used as follows: for $1 \leq p < \infty$,

$$\|f\|_{L^p(\mathbb{F}_q^d, dx)} = \left(\frac{1}{q^d} \sum_{x \in \mathbb{F}_q^d} |f(x)|^p \right)^{1/p} \quad \text{and} \quad \|g\|_{L^p(\mathbb{F}_q^d, dm)} = \left(\sum_{m \in \mathbb{F}_q^d} |f(m)|^p \right)^{1/p}.$$

Also recall that $\|f\|_{L^\infty(\mathbb{F}_q^d, dx)} = \max_{x \in \mathbb{F}_q^d} |f(x)|$ and $\|g\|_{L^\infty(\mathbb{F}_q^d, dm)} = \max_{m \in \mathbb{F}_q^d} |g(m)|$. The cone $C \subset (\mathbb{F}_q^d, dx)$, $d \geq 3$, is defined by the set

$$(1.2) \quad C = \{x \in \mathbb{F}_q^d : x_1^2 + x_2^2 + \cdots + x_{d-2}^2 = x_{d-1}x_d\}.$$

Mockenhaupt and Tao [20] gave the complete answer to the restriction problem for the cone C in three dimensions. We endow the cone C with the normalized surface measure σ which is defined by the relation

$$\int_C f(x) d\sigma(x) := \frac{1}{|C|} \sum_{x \in C} f(x),$$

where $|C|$ denotes the cardinality of the cone $C \subset \mathbb{F}_q^d$, and $f : (\mathbb{F}_q^d, dx) \rightarrow \mathbb{C}$. In other words, the mass of each point of the cone C is considered as $1/|C|$.

Remark 1.1. Since $d\sigma(x) = \frac{q^d}{|C|} C(x) dx$, the normalized surface measure σ on the cone C can be identified with a function $\frac{q^d}{|C|} C(x)$ on (\mathbb{F}_q^d, dx) , where we write $C(x)$ for the characteristic function χ_C on the cone C . Namely, we shall identify a set E with the characteristic function χ_E , which allows us to use simple notation.

Let g be a complex-valued function on (\mathbb{F}_q^d, dm) . Then \widehat{g} , the Fourier transform of g is defined on the dual space (\mathbb{F}_q^d, dx) as follows:

$$\widehat{g}(x) = \int_{\mathbb{F}_q^d} \chi(-m \cdot x) g(m) dm = \sum_{m \in \mathbb{F}_q^d} \chi(-m \cdot x) g(m),$$

where χ denotes a nontrivial additive character of \mathbb{F}_q and $m \cdot x$ is the usual dot-product notation. Given a function $f : (\mathbb{F}_q^d, dx) \rightarrow \mathbb{C}$, the inverse Fourier transform of f , denoted by f^\vee , is defined on (\mathbb{F}_q^d, dm) and it takes the following form

$$f^\vee(m) = \int_{\mathbb{F}_q^d} \chi(m \cdot x) f(x) dx = \frac{1}{q^d} \sum_{x \in \mathbb{F}_q^d} \chi(m \cdot x) f(x).$$

We stress that the choice of χ does not change our results in this paper as long as χ is a nontrivial additive character of \mathbb{F}_q . Recall that the orthogonality relation of χ states that

$$\sum_{x \in \mathbb{F}_q^d} \chi(m \cdot x) = \begin{cases} 0 & \text{if } m \neq (0, \dots, 0) \\ q^d & \text{if } m = (0, \dots, 0), \end{cases}$$

Observe that the Plancherel theorem states $\|f^\vee\|_{L^2(\mathbb{F}_q^d, dm)} = \|f\|_{L^2(\mathbb{F}_q^d, dx)}$, which yields

$$\sum_{m \in \mathbb{F}_q^d} |f^\vee(m)|^2 = \frac{1}{q^d} \sum_{x \in \mathbb{F}_q^d} |f(x)|^2.$$

1.2. Restricted averaging problem and statement of main result. Given two functions $f, h : (\mathbb{F}_q^d, dx) \rightarrow \mathbb{C}$, the convolution function $f * h$ is defined on (\mathbb{F}_q^d, dx) as follows:

$$f * h(y) = \int_{\mathbb{F}_q^d} f(y - x) h(x) dx = \frac{1}{q^d} \sum_{x \in \mathbb{F}_q^d} f(y - x) h(x) \quad \text{for } y \in (\mathbb{F}_q^d, dx).$$

It clearly follows that $(f * h)^\vee(m) = f^\vee(m) h^\vee(m)$ for $m \in (\mathbb{F}_q^d, dm)$. Replacing the function h by the normalized surface measure μ on an algebraic variety $V \subset (\mathbb{F}_q^d, dx)$, the averaging operator A can be defined by

$$Af(y) = f * \mu(y) = \int_V f(y - x) d\mu(x) := \frac{1}{|V|} \sum_{x \in V} f(y - x),$$

where both f and Af are defined on (\mathbb{F}_q^d, dx) . In the finite field setting, Carbery-Stones-Wright [5] initially studied the $L^p \rightarrow L^r$ estimates for the averaging operator. Much attention has been given to this problem (for example, see [12, 14, 16]).

As a variant of the averaging operator A over $(V, \mu) \subset (\mathbb{F}_q^d, dx)$, a restricted averaging operator A_V to V is defined by restricting $Af = f * \mu$ to the variety V . Namely, we have $A_V f = Af|_V = f * d\mu|_V$. Then the restricted averaging problem is to determine $1 \leq p, r \leq \infty$ such that the following restricted averaging inequality holds:

$$(1.3) \quad \|A_V f\|_{L^r(V, \mu)} \leq C \|f\|_{L^p(\mathbb{F}_q^d, dx)},$$

where the constant $C > 0$ is independent of the functions f and the size of the underlying finite field \mathbb{F}_q . This problem was proposed in [13] where the sharp restricted averaging inequality was established in the case when the variety V is any curve on plane. Such a sharp result was obtained by a direct application of the complete solution to the Fourier restriction problem for curves in two dimensions. The authors in [15] observed from the Fourier decay estimate that the optimal restricted averaging inequalities can be obtained if the variety $V \subset (\mathbb{F}_q^d, dx)$ satisfies the following two conditions:

$$(1.4) \quad |V| \sim q^{d-1} \quad \text{and} \quad \max_{m \neq (0, \dots, 0)} |V^\vee(m)| \lesssim q^{-(d+1)/2}.$$

Here, and throughout this paper, we write $E(x)$ for the characteristic function χ_E on the set $E \subset \mathbb{F}_q^d$. Also recall that $X \lesssim Y$ is used to denote that there exists $C > 0$ independent of $q = |\mathbb{F}_q|$ such that $X \leq CY$. In addition, $X \sim Y$ means $X \lesssim Y \lesssim X$. We shall write $A_V(p \rightarrow r) \lesssim 1$ if the restricted averaging inequality (1.3) holds.

A variety $V \subset (\mathbb{F}_q^d, dx)$ satisfying the conditions (1.4) is called a regular variety. Typical regular varieties are the paraboloids and the spheres with nonzero radius. When the restricted operator A_V is related to a non-regular variety $V \subset (\mathbb{F}_q^d, dx)$, it may not be a simple problem to find the sharp restricted averaging inequality, because the optimal results can not be obtained by simply applying the Fourier decay estimate. Therefore, it would be interesting to prove sharp restricted averaging inequalities for non-regular varieties. The cone $C \subset (\mathbb{F}_q^d, dx)$ defined as in (1.2) has unusual structures in that it is not a regular variety in even dimensions $d \geq 4$, but it is a regular variety in odd dimensions $d \geq 3$ (see Corollary 4.3 in [15]). For this reason, we are interested in establishing the sharp restricted averaging problem on cones $C \subset (\mathbb{F}_q^d, dx)$ in even dimensions $d \geq 4$. The necessary conditions for the boundedness of $A_C(p \rightarrow r)$ were given in [15]. For example, the lemma below follows immediately from Lemma 2.1 in [15].

Lemma 1.2. *Let σ denote the normalized surface measure on the cone $C \subset (\mathbb{F}_q^d, dx)$. Suppose that the restricted averaging estimate*

$$\|A_C f\|_{L^r(C, \sigma)} \lesssim \|f\|_{L^p(\mathbb{F}_q^d, dx)}$$

holds for all function f on (\mathbb{F}_q^d, dx) . Then the following two statements are true:

- (1) *If the cone C does not contain any subspace H with $|H| > q^{\frac{d-1}{2}}$, then $\left(\frac{1}{p}, \frac{1}{r}\right)$ must lie on the convex hull of points $(0, 0)$, $(0, 1)$, $\left(\frac{d-1}{d}, 1\right)$ and $P_0 := \left(\frac{d-1}{d}, \frac{1}{d}\right)$.*
- (2) *If the cone C contains a $d/2$ -dimensional subspace H , then $\left(\frac{1}{p}, \frac{1}{r}\right)$ lies on the convex hull of points $(0, 0)$, $(0, 1)$, $\left(\frac{d-1}{d}, 1\right)$, $P_1 := \left(\frac{d-1}{d}, \frac{1}{d-2}\right)$ and $P_2 := \left(\frac{d^2-3d+2}{d^2-2d+2}, \frac{d-2}{d^2-2d+2}\right)$.*

From the nesting property of norms and the interpolation with the trivial $L^\infty \rightarrow L^\infty$ estimate, to prove that the necessary conditions are in fact sufficient, it suffices to obtain the critical point P_0 . In addition, to prove the optimal results in the case when the cone C contains a $d/2$ -dimensional subspace, it will be enough to obtain the critical points P_1 and P_2 . In fact, when $d \geq 3$ is odd, the critical point P_0 was obtained in [15], which gives the complete answer to the restricted averaging problem for cones in odd dimensions. On the other hand, when $d \geq 4$ is even, it is in general impossible to obtain the point P_0 , because the cone C may contain a $d/2$ -dimensional subspace. As we shall see, the cone $C \subset (\mathbb{F}_q^d, dx)$ contains a $d/2$ -dimensional subspace if $d = 4k + 2$ for $k \in \mathbb{N}$, or if $-1 \in \mathbb{F}_q$ is a square number and $d \geq 4$ is even. In this case, to settle the restricted averaging problem for cones, it suffices to obtain the critical points P_1 and P_2 . In this paper, we shall establish the critical points except for dimension four. As a consequence, we give complete answers to the restricted averaging problems for cones in even dimensions $d \geq 6$ in the case when the cone C contains a $d/2$ -dimensional subspace. More precisely, our main theorem is as follows:

Theorem 1.3. *Let A_C be the restricted averaging operator associated with the cone $C \subset (\mathbb{F}_q^d, dx)$ defined as in (1.2). Suppose that σ denotes the normalized surface measure on the cone C . Then, if $d \geq 6$ is even, we have*

$$(1.5) \quad \|A_C f\|_{L^{d-2}(C, \sigma)} \lesssim \|f\|_{L^{\frac{d}{d-1}}(\mathbb{F}_q^d, dx)}$$

and if $d \geq 4$ is even, then we have

$$(1.6) \quad \|A_C f\|_{L^{\frac{d^2-2d+2}{d-2}}(C, \sigma)} \lesssim \|f\|_{L^{\frac{d^2-2d+2}{d^2-3d+2}}(\mathbb{F}_q^d, dx)}$$

In [15], it was proved that the inequality (1.5) holds if $d \geq 4$ is even and the test functions f are characteristic functions on (\mathbb{F}_q^d, dx) . It was also proved in [15] that the dual estimate of the inequality (1.6) holds for all characteristic test functions on the cone C in even dimensions $d \geq 4$. Hence, Theorem 1.3 provides the improved endpoint estimates in even dimensions $d \geq 6$. The estimate (1.6) gives a partial improvement in four dimensions.

1.3. Remark on sharpness of Theorem 1.3. As mentioned before, we see from Theorem 1.3 and Lemma 1.2 that if $d \geq 6$ is even and the cone $C \subset (\mathbb{F}_q^d, dx)$ contains a $d/2$ -dimensional subspace, then $A_C(p \rightarrow r) \lesssim 1$ if and only if $(\frac{1}{p}, \frac{1}{r})$ is contained in the convex hull of the points $(0, 0), (0, 1), (\frac{d-1}{d}, 1), P_1 := (\frac{d-1}{d}, \frac{1}{d-2})$ and $P_2 := (\frac{d^2-3d+2}{d^2-2d+2}, \frac{d-2}{d^2-2d+2})$. Let η denote the quadratic character of \mathbb{F}_q^* . In addition, assume that H denotes a maximal subspace contained in the cone $C \subset (\mathbb{F}_q^d, dx)$. It is well known that for even dimensions $d \geq 4$ we have

$$(1.7) \quad |H| = \begin{cases} q^{\frac{d}{2}} & \text{if } \eta(-1) = (\eta(-1))^{\frac{d}{2}} \\ q^{\frac{d-2}{2}} & \text{if } \eta(-1) = -(\eta(-1))^{\frac{d}{2}} \end{cases}$$

(for example, see Lemma 2.1 in [24]). Thus, if $d = 4k + 2$ for $k \in \mathbb{N}$, or $-1 \in \mathbb{F}_q$ is a square number and $d \geq 4$ is even, then the cone C contains a subspace H with $|H| = q^{\frac{d}{2}}$. In conclusion, Theorem 1.3 provides the complete mapping properties of the restricted averaging operator A_C in the case when $d = 4k + 2$ for $k \in \mathbb{N}$, or $-1 \in \mathbb{F}_q$ is a square number and $d \geq 6$ is even.

Remark 1.4. Notice from Theorem 1.3 that to settle the restricted averaging problem for the cone C in the case when $d = 4$ and $-1 \in \mathbb{F}_q$ is a square number, we only need to prove the inequality (1.5) for $d = 4$. However, it looks a hard problem and we leave this as an open question.

From (1.7) we see that if $-1 \in \mathbb{F}_q$ is not a square number and $d = 4k$ for $k \in \mathbb{N}$, then $q^{\frac{d-2}{2}}$ is the cardinality of a maximal subspace lying in the cone $C \subset (\mathbb{F}_q^d, dx)$. Combining this fact with the first conclusion of Lemma 1.2, we may conjecture the following.

Conjecture 1.5. *Let $C \subset (\mathbb{F}_q^d, dx)$ be the cone. Assume that $d = 4k$ for $k \in \mathbb{N}$, and $-1 \in \mathbb{F}_q$ is not a square number. Then we have $A_C(p \rightarrow r) \lesssim 1$ if and only if $(\frac{1}{p}, \frac{1}{r})$ lies on the convex hull of points $(0, 0), (0, 1), (\frac{d-1}{d}, 1)$ and $P_0 := (\frac{d-1}{d}, \frac{1}{d})$.*

As seen before, in order to establish this conjecture, it will be enough to obtain the critical point P_0 .

1.4. Contents of the remain parts of this paper. The remain parts of this paper will be organized as follows. In Section 2, we introduce preliminary key lemmas which play a crucial role in proving Theorem 1.3. The proof of the inequalities (1.5) and (1.6) in Theorem 1.3 will be given in Sections 3 and 4, respectively.

2. PRELIMINARY LEMMAS

In this section, we collect several lemmas most of which are implicitly contained in [15]. Let us denote by A_C^* the adjoint operator of the restricted averaging operator A_C to the cone $C \subset (\mathbb{F}_q^d, dx)$. Since $A_C f = f * \sigma|_C$, it follows that

$$\langle A_C f, h \rangle_{L^2(C, \sigma)} = \langle f, A_C^* h \rangle_{L^2(\mathbb{F}_q^d, dx)},$$

where we recall that σ is the normalized surface measure on the cone C . From this, we see that the adjoint operator A_C^* is given by

$$A_C^* h(y) = \frac{q^d}{|C|^2} \sum_{x \in C} C(x - y) h(x)$$

where $h : (C, \sigma) \rightarrow \mathbb{C}$ and $y \in (\mathbb{F}_q^d, dx)$. Since $C = -C$, we can alternatively write that for all functions $h : (\mathbb{F}_q^d, dx) \rightarrow \mathbb{C}$ with $h(x) = 0$ for $x \in \mathbb{F}_q^d \setminus C$.

$$(2.1) \quad A_C^* h = \frac{q^{2d}}{|C|^2} (hC) * C = \frac{q^{2d}}{|C|^2} h * C.$$

We aim to find the exponents $1 \leq p, r \leq \infty$ such that

$$\|A_C f\|_{L^r(C, \sigma)} \lesssim \|f\|_{L^p(\mathbb{F}_q^d, dx)}.$$

By duality, this equality is same as the following inequality

$$\|A_C^* h\|_{L^{p'}(\mathbb{F}_q^d, dx)} \lesssim \|h\|_{L^{r'}(C, \sigma)},$$

where $p' = p/(p-1)$ and $r' = r/(r-1)$.

2.1. Decomposition of the restricted averaging operator. Define a function K on (\mathbb{F}_q^d, dm) by

$$(2.2) \quad K(m) = \sigma^\vee(m) - \delta_0(m),$$

where $\delta_0(m) = 1$ for $m = (0, \dots, 0)$ and 0 otherwise. Then we can write $\sigma(x) = \widehat{K}(x) + \widehat{\delta}_0(x) = \widehat{K}(x) + 1$. Thus, the restricted averaging operator A_C to the cone C can be decomposed by

$$(2.3) \quad A_C f = f * \sigma = f * 1 + f * \widehat{K}.$$

Observe from the definition of K that $K(m) = \sigma^\vee(m)$ for $m \neq (0, \dots, 0)$ and $K(0, \dots, 0) = 0$. Then the following lemma is a direct result from Corollary 4.4 in [15].

Lemma 2.1. *Let σ be the normalized surface measure on the cone $C \subset (\mathbb{F}_q^d, dx)$ defined as in (1.2). Define $K(m) = \sigma^\vee(m) - \delta_0(m)$ for $m \in (\mathbb{F}_q^d, dm)$ and $\Gamma(\xi) = \xi_1^2 + \xi_2^2 + \dots + \xi_{d-2}^2 - 4\xi_{d-1}\xi_d$ for $\xi = (\xi_1, \dots, \xi_d) \in \mathbb{F}_q^d$. If the dimension $d \geq 4$ is even and $m \neq (0, \dots, 0) \in \mathbb{F}_q^d$, then we have*

$$|K(m)| = |\sigma^\vee(m)| \sim \begin{cases} q^{-\frac{(d-2)}{2}} & \text{for } \Gamma(m) = 0 \\ q^{-\frac{d}{2}} & \text{for } \Gamma(m) \neq 0. \end{cases}$$

In addition, we have $K(0, \dots, 0) = 0$.

The following Fourier restriction estimate was given in Lemma 3.1 in [15].

Lemma 2.2. *Let σ be the normalized surface measure on the cone $C \subset (\mathbb{F}_q^d, dx)$. Then we have*

$$\|\widehat{g}\|_{L^2(C, \sigma)} \lesssim q^{\frac{1}{2}} \|g\|_{L^2(\mathbb{F}_q^d, dm)} \quad \text{for all } g : (\mathbb{F}_q^d, dm) \rightarrow \mathbb{C}.$$

Proof. By duality, it is enough to prove the following extension estimate:

$$\|(f\sigma)^\vee\|_{L^2(\mathbb{F}_q^d, dm)} \sim q^{\frac{1}{2}} \|f\|_{L^2(C, \sigma)} \quad \text{for all } f : C \rightarrow \mathbb{C}.$$

Since $\sigma(x) = \frac{q^d}{|C|} C(x)$ and $|C| \sim q^{d-1}$, the Plancherel theorem yields

$$\begin{aligned} \|(f\sigma)^\vee\|_{L^2(\mathbb{F}_q^d, dm)} &= \frac{q^d}{|C|} \|(fC)^\vee\|_{L^2(\mathbb{F}_q^d, dm)} = \frac{q^d}{|C|} \|fC\|_{L^2(\mathbb{F}_q^d, dx)} \\ &= \frac{q^{d/2}}{|C|^{1/2}} \|f\|_{L^2(C, \sigma)} \sim q^{\frac{1}{2}} \|f\|_{L^2(C, \sigma)}. \end{aligned}$$

□

The following lemma was also given in Lemma 4.5 in [15].

Lemma 2.3. *Let $C^* = \{m \in \mathbb{F}_q^d : \Gamma(m) = 0\}$. If the dimension, $d \geq 4$ is even, then we have*

$$\sum_{m \in C^*} |E^\vee(m)|^2 \lesssim q^{-d-1}|E| + q^{-\frac{3d}{2}}|E|^2 \quad \text{for all } E \subset (\mathbb{F}_q^d, dx).$$

We shall invoke the following result.

Lemma 2.4. *Let σ be the normalized surface measure on the cone $C \subset (\mathbb{F}_q^d, dx)$. Then if $d \geq 4$ is even, the estimate*

$$\sum_{m \in \mathbb{F}_q^d \setminus \{(0, \dots, 0)\}} |E^\vee(m) \sigma^\vee(m)|^2 \lesssim \min \left\{ q^{-2d+2}|E|, q^{-2d+1}|E| + q^{\frac{-5d+4}{2}}|E|^2 \right\}$$

holds for all sets $E \subset (\mathbb{F}_q^d, dx)$.

Proof. Let Γ be the function defined as in the statement of Lemma 2.1. We write

$$\sum_{m \neq (0, \dots, 0)} |E^\vee(m) \sigma^\vee(m)|^2 = \sum_{\substack{m \neq (0, \dots, 0) \\ \Gamma(m) \neq 0}} |E^\vee(m) \sigma^\vee(m)|^2 + \sum_{\substack{m \neq (0, \dots, 0) \\ \Gamma(m) = 0}} |E^\vee(m) \sigma^\vee(m)|^2.$$

Applying Lemma 2.1, Lemma 2.3, and the Plancherel theorem, we see that

$$\begin{aligned} \sum_{m \neq (0, \dots, 0)} |E^\vee(m) \sigma^\vee(m)|^2 &\lesssim q^{-d} \sum_{m \in \mathbb{F}_q^d} |E^\vee(m)|^2 + q^{-d+2} \sum_{\Gamma(m)=0} |E^\vee(m)|^2 \\ &\lesssim q^{-d} q^{-d} |E| + \left(q^{-2d+1} |E| + q^{\frac{-5d+4}{2}} |E|^2 \right) \\ &\lesssim q^{-2d+1} |E| + q^{\frac{-5d+4}{2}} |E|^2. \end{aligned}$$

On the other hand, we also see from Lemma 2.1 and the Plancherel theorem that

$$\sum_{m \neq (0, \dots, 0)} |E^\vee(m) \sigma^\vee(m)|^2 \lesssim q^{-d+2} \sum_{m \in \mathbb{F}_q^d} |E^\vee(m)|^2 = q^{-2d+2} |E|.$$

Putting all estimates together, we obtain the statement of the lemma. \square

The following lemma will play an important role in deriving our main result.

Lemma 2.5. *Let K be defined as in (2.2). If the dimension $d \geq 4$ is even, then the estimates*

$$(2.4) \quad \|E * \widehat{K}\|_{L^\infty(C, \sigma)} \lesssim \frac{|E|}{q^{d-1}}$$

and

$$(2.5) \quad \|E * \widehat{K}\|_{L^2(C, \sigma)} \lesssim \min \left\{ q^{\frac{-2d+3}{2}} |E|^{\frac{1}{2}}, q^{-d+1} |E|^{\frac{1}{2}} + q^{\frac{-5d+6}{4}} |E| \right\}$$

hold for all $E \subset (\mathbb{F}_q^d, dx)$.

Proof. To prove the inequality (2.4), observe from Remark 1.1 that

$$\max_{y \in \mathbb{F}_q^d} |\widehat{K}(y)| = \max_{y \in \mathbb{F}_q^d} |\sigma(y) - 1| = \max_{y \in \mathbb{F}_q^d} \left| \frac{q^d C(y)}{|C|} - 1 \right| \leq \frac{q^d}{|C|} \sim q,$$

where we used that $|C| \sim q^{d-1}$. Then it follows that for any $x \in C$,

$$|E * \widehat{K}(x)| \leq \left(\max_{y \in \mathbb{F}_q^d} |\widehat{K}(y)| \right) \frac{1}{q^d} \sum_{y \in \mathbb{F}_q^d} |E(x-y)| \sim \frac{|E|}{q^{d-1}},$$

and we obtain the inequality (2.4). Next, in order to prove the inequality (2.5) holds, it will be enough to show that

$$(2.6) \quad \|E * \widehat{K}\|_{L^2(C, \sigma)}^2 \lesssim \min \left\{ q^{-2d+3} |E|, q^{-2d+2} |E| + q^{\frac{-5d+6}{2}} |E|^2 \right\}.$$

Since $E * \widehat{K} = \widehat{E^\vee K}$, we see from Lemma 2.2 that

$$\|E * \widehat{K}\|_{L^2(C, \sigma)}^2 = \|\widehat{E^\vee K}\|_{L^2(C, \sigma)}^2 \leq q \|E^\vee K\|_{L^2(\mathbb{F}_q^d, dm)}^2.$$

By the definition of K , the right-hand side is written by

$$q \sum_{m \neq (0, \dots, 0)} |E^\vee(m) \sigma^\vee(m)|^2.$$

Applying Lemma 2.4 to this estimate, we obtain the inequality (2.6). Thus, we complete the proof of the inequality (2.5). \square

The following result is much weaker than (2.5) of Theorem 2.5, but it is useful to apply in practice.

Corollary 2.6. If $d \geq 4$ is even, then we have

$$\|E * \widehat{K}\|_{L^2(C, \sigma)} \lesssim q^{-d+1} |E|^{\frac{d+2}{2d}}$$

for all $E \subset (\mathbb{F}_q^d, dx)$.

Proof. Notice that the estimate (2.5) of Lemma 2.5 implies that if $d \geq 4$ is even, the estimate

$$(2.7) \quad \|E * \widehat{K}\|_{L^2(C, \sigma)} \lesssim \begin{cases} q^{\frac{-2d+3}{2}} |E|^{\frac{1}{2}} & \text{if } q^{\frac{d}{2}} \leq |E| \leq q^d \\ q^{\frac{-5d+6}{4}} |E| & \text{if } q^{\frac{d-2}{2}} \leq |E| \leq q^{\frac{d}{2}} \\ q^{-d+1} |E|^{\frac{1}{2}} & \text{if } 1 \leq |E| \leq q^{\frac{d-2}{2}} \end{cases}$$

holds for all $E \subset (\mathbb{F}_q^d, dx)$, which in turn implies the conclusion of the corollary. \square

The following result will be used to deduce the estimate (1.5) of Theorem 1.3.

Lemma 2.7. *If the dimension $d \geq 6$ is even, the estimate*

$$\|E * \widehat{K}\|_{L^{\frac{d-2}{2}}(C, \sigma)} \lesssim q^{-d+1} |E|^{\frac{d-2}{d}}$$

holds for all $E \subset (\mathbb{F}_q^d, dx)$.

Proof. Since $2 \leq \frac{d-2}{2} < \infty$ for $d \geq 6$, the statement follows immediately by interpolating the estimate (2.4) of Lemma 2.5 and the conclusion of Corollary 2.6. \square

2.2. Decomposition of the dual restricted averaging operator. We shall decompose the dual operator A_C^* defined as in (2.1). We define a function M on (\mathbb{F}_q^d, dm) by

$$(2.8) \quad M(m) = C^\vee(m) - \frac{|C|}{q^d} \delta_0(m) \quad \text{for } m \in (\mathbb{F}_q^d, dm).$$

Then for each $x \in (\mathbb{F}_q^d, dx)$ we can write

$$(2.9) \quad C(x) = \widehat{M}(x) + \frac{|C|}{q^d} \widehat{\delta}_0(x) = \widehat{M}(x) + \frac{|C|}{q^d}.$$

Namely, the characteristic function on the cone C is same as the function $\widehat{M} + \frac{|C|}{q^d}$. Recall from (2.1) that we can write

$$A_C^* h = \frac{q^{2d}}{|C|^2} h * C$$

where h is a function supported on C . Thus, A_C^* can be decomposed as

$$A_C^* h = \frac{q^{2d}}{|C|^2} h * \widehat{M} + \frac{q^d}{|C|} h * 1.$$

The following lemma plays a crucial role in proving the inequality (1.6) of Theorem 1.3.

Lemma 2.8. *Let M be the function defined as in (2.8). If the dimension, $d \geq 4$, is even, then the estimates*

$$(2.10) \quad \|F * \widehat{M}\|_{L^\infty(\mathbb{F}_q^d, dx)} \lesssim \frac{|F|}{q^d}$$

and

$$(2.11) \quad \|F * \widehat{M}\|_{L^2(\mathbb{F}_q^d, dx)} \lesssim \min \left\{ q^{-d} |E|^{\frac{1}{2}}, q^{\frac{-2d-1}{2}} |E|^{\frac{1}{2}} + q^{\frac{-5d}{4}} |E| \right\}$$

hold for all $F \subset (C, \sigma)$.

Proof. To prove the inequality (2.10), we notice from Young's inequality for convolutions that

$$\|F * \widehat{M}\|_{L^\infty(\mathbb{F}_q^d, dx)} \leq \|F\|_{L^1(\mathbb{F}_q^d, dx)} \|\widehat{M}\|_{L^\infty(\mathbb{F}_q^d, dx)} = \frac{|F|}{q^d} \|\widehat{M}\|_{L^\infty(\mathbb{F}_q^d, dx)}.$$

Since $|C| \sim q^{d-1}$, it is clear from (2.9) that $\|\widehat{M}\|_{L^\infty(\mathbb{F}_q^d, dx)} \lesssim 1$. Thus, the inequality (2.10) holds. Next, we shall prove the inequality (2.11). Squaring the both sides of the inequality (2.11), it suffices to show that

$$(2.12) \quad \|F * \widehat{M}\|_{L^2(\mathbb{F}_q^d, dx)}^2 \lesssim \min \left\{ q^{-2d} |E|, q^{-2d-1} |E| + q^{\frac{-5d}{2}} |E|^2 \right\} \quad \text{for all } F \subset C.$$

By the Plancherel theorem, it follows that

$$\|F * \widehat{M}\|_{L^2(\mathbb{F}_q^d, dx)}^2 = \|F^\vee M\|_{L^2(\mathbb{F}_q^d, dm)}^2 = \sum_{m \in \mathbb{F}_q^d} |F^\vee(m) M(m)|^2.$$

By the definition of M in (2.8), it is clear that $M(m) = C^\vee(m)$ for $m \neq (0, \dots, 0)$ and $M(0, \dots, 0) = 0$. Also recall that the normalized surface measure σ on the cone

C can be identified with a function $\sigma(x) = \frac{q^d}{|C|}C(x) \sim qC(x)$. It follows that

$$\|F * \widehat{M}\|_{L^2(\mathbb{F}_q^d, dx)}^2 = \sum_{m \neq (0, \dots, 0)} |F^\vee(m) C^\vee(m)|^2 \sim q^{-2} \sum_{m \neq (0, \dots, 0)} |F^\vee(m) \sigma^\vee(m)|^2.$$

Then the estimate (2.12) is obtained by using Lemma 2.4. Thus, the proof is complete. \square

By a direct computation, the following result is obtained from (2.11) of Lemma 2.8.

Corollary 2.9. Let $d \geq 4$ be even. Then the estimate

$$(2.13) \quad \|F * \widehat{M}\|_{L^2(\mathbb{F}_q^d, dx)} \lesssim \begin{cases} q^{-d} |F|^{\frac{1}{2}} & \text{if } q^{\frac{d}{2}} \leq |F| \lesssim q^{d-1} \\ q^{-\frac{5d}{4}} |F| & \text{if } q^{\frac{d-2}{2}} \leq |F| \leq q^{\frac{d}{2}} \\ q^{-\frac{2d-1}{2}} |F|^{\frac{1}{2}} & \text{if } 1 \leq |F| \leq q^{\frac{d-2}{2}} \end{cases}$$

holds for all $F \subset (C, \sigma)$.

We shall need the following estimates.

Lemma 2.10. If $d \geq 6$ is even, then the following estimate holds for all $F \subset (C, \sigma)$:

$$(2.14) \quad \|F * \widehat{M}\|_{L^{\frac{d^2-2d+2}{2d}}(\mathbb{F}_q^d, dx)} \lesssim \frac{|F|^{\frac{d^2-4d+6}{d^2-2d+2}}}{q^{\frac{d^3-2d^2+4d}{d^2-2d+2}}}.$$

On the other hand, in the dimension four, the estimate

$$(2.15) \quad \|F * \widehat{M}\|_{L^{\frac{10}{3}}(\mathbb{F}_q^4, dx)} \lesssim \begin{cases} q^{-4} |F|^{\frac{7}{10}} & \text{if } q^2 \leq |F| \lesssim q^3 \\ q^{-\frac{23}{5}} |F| & \text{if } q \leq |F| \leq q^2 \\ q^{-\frac{43}{10}} |F|^{\frac{7}{10}} & \text{if } 1 \leq |F| \leq q \end{cases}$$

holds for all $F \subset (C, \sigma)$.

Proof. First, let us prove the estimate (2.14). By a direct comparison, we see that the estimate (2.13) of Corollary 2.9 implies that if $d \geq 4$ is even, then

$$(2.16) \quad \|F * \widehat{M}\|_{L^2(\mathbb{F}_q^d, dx)} \lesssim q^{-\frac{2d-1}{2}} |F|^{\frac{d+2}{2d}} \quad \text{for all } F \subset (C, \sigma).$$

Recall from (2.10) of Lemma 2.8 that if $d \geq 4$ is even, then

$$(2.17) \quad \|F * \widehat{M}\|_{L^\infty(\mathbb{F}_q^d, dx)} \lesssim q^{-d} |F| \quad \text{for all } F \subset (C, \sigma).$$

Since $2 < \frac{d^2-2d+2}{2d} < \infty$ for $d \geq 6$, the estimate (2.14) follows by interpolating (2.16) and (2.17).

Next, to prove the estimate (2.15), notice that if $d = 4$, then (2.10) of Lemma 2.8 and (2.13) of Corollary 2.9 yield the following two estimates: for every $F \subset (C, \sigma)$

$$\|F * \widehat{M}\|_{L^\infty(\mathbb{F}_q^4, dx)} \lesssim q^{-4} |F|$$

and

$$(2.18) \quad \|F * \widehat{M}\|_{L^2(\mathbb{F}_q^4, dx)} \lesssim \begin{cases} q^{-4} |F|^{\frac{1}{2}} & \text{if } q^2 \leq |F| \lesssim q^3 \\ q^{-5} |F| & \text{if } q \leq |F| \leq q^2 \\ q^{-\frac{9}{2}} |F|^{\frac{1}{2}} & \text{if } 1 \leq |F| \leq q. \end{cases}$$

Since $2 < \frac{10}{3} < \infty$, the estimate (2.15) of Lemma 2.10 follows by interpolating the above two estimates. \square

3. THE PROOF OF THE INEQUALITY (1.5) IN THEOREM 1.3

In this section, we restate and prove the first part of Theorem 1.3.

Theorem 3.1. *Let A_C be the restricted averaging operator associated with the cone $C \subset (\mathbb{F}_q^d, dx)$ defined as in (1.2). Suppose that σ denotes the normalized surface measure on the cone C . Then, if $d \geq 6$ is even, the estimate*

$$\|A_C f\|_{L^{d-2}(C, \sigma)} \lesssim \|f\|_{L^{\frac{d}{d-1}}(\mathbb{F}_q^d, dx)}$$

holds for all functions f on \mathbb{F}_q^d .

Proof. We aim to show that the estimate

$$\|f * \sigma\|_{L^{d-2}(C, \sigma)} \lesssim \|f\|_{L^{\frac{d}{d-1}}(\mathbb{F}_q^d, dx)} = q^{-d+1} \left(\sum_{x \in \mathbb{F}_q^d} |f(x)|^{\frac{d}{d-1}} \right)^{\frac{d-1}{d}}$$

holds for all functions f on \mathbb{F}_q^d . Without loss of generality, we may assume that f is a non-negative real-valued function and

$$(3.1) \quad \sum_{x \in \mathbb{F}_q^d} f(x)^{\frac{d}{d-1}} = 1.$$

Then $\|f\|_{\infty} \leq 1$ and so we may assume that f is written by a step function

$$(3.2) \quad f(x) = \sum_{i=0}^{\infty} 2^{-i} E_i(x),$$

where E_i 's are pairwise disjoint subsets of \mathbb{F}_q^d . Combining (3.1) with (3.2), we also assume that

$$(3.3) \quad \sum_{j=0}^{\infty} 2^{-\frac{dj}{d-1}} |E_j| = 1 \quad \text{and so } |E_j| \leq 2^{\frac{dj}{d-1}} \text{ for all } j = 0, 1, \dots$$

Thus, to complete the proof we only need to show that the estimate

$$\|f * \sigma\|_{L^{d-2}(C, \sigma)} \lesssim q^{-d+1}$$

holds for all functions f on \mathbb{F}_q^d satisfying the assumptions (3.1), (3.2), (3.3). As seen in (2.3), we can write $f * \sigma = f * 1 + f * \widehat{K}$, and thus our problem is reduced to showing that the following two estimates hold:

$$(3.4) \quad \|f * 1\|_{L^{d-2}(C, \sigma)} \lesssim q^{-d+1}$$

and

$$(3.5) \quad \|f * \widehat{K}\|_{L^{d-2}(C, \sigma)} \lesssim q^{-d+1},$$

where the function K on (\mathbb{F}_q^d, dm) is defined as in (2.2). Since $\max_{x \in C} |f * 1(x)| \leq \|f\|_{L^1(\mathbb{F}_q^d, dx)}$, the estimate (3.4) can be obtained by observing

$$\|f * 1\|_{L^{d-2}(C, \sigma)} \leq \|f\|_{L^1(\mathbb{F}_q^d, dx)} \leq \|f\|_{L^{\frac{d}{d-1}}(\mathbb{F}_q^d, dx)} = q^{-d+1},$$

where we used the assumption (3.1). It remains to prove the estimate (3.5) which is in turn written by

$$(3.6) \quad A := q^{2d-2} \|(f * \widehat{K})(f * \widehat{K})\|_{L^{\frac{d-2}{2}}(C, \sigma)} \lesssim 1.$$

Using (3.2), we see that

$$\begin{aligned} A &\leq q^{2d-2} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} 2^{-k-j} \|(E_k * \widehat{K})(E_j * \widehat{K})\|_{L^{\frac{d-2}{2}}(C, \sigma)} \\ &\sim q^{2d-2} \sum_{k=0}^{\infty} \sum_{j=k}^{\infty} 2^{-k-j} \|(E_k * \widehat{K})(E_j * \widehat{K})\|_{L^{\frac{d-2}{2}}(C, \sigma)}, \end{aligned}$$

where the last line is obtained by the symmetry of k, j . Using (2.4) of Lemma 2.5 and Lemma 2.7, we see that

$$\begin{aligned} A &\lesssim q^{2d-2} \sum_{k=0}^{\infty} \sum_{j=k}^{\infty} 2^{-k-j} \|E_k * \widehat{K}\|_{L^{\infty}(C, \sigma)} \|E_j * \widehat{K}\|_{L^{\frac{d-2}{2}}(C, \sigma)} \\ &\lesssim \sum_{k=0}^{\infty} \sum_{j=k}^{\infty} 2^{-k-j} |E_k| |E_j|^{\frac{d-2}{d}}. \end{aligned}$$

By (3.3), we conclude that

$$\begin{aligned} A &\lesssim \sum_{k=0}^{\infty} \sum_{j=k}^{\infty} 2^{-k-j} |E_k| 2^{\frac{(d-2)j}{d-1}} = \sum_{k=0}^{\infty} 2^{-k} |E_k| \left(\sum_{j=k}^{\infty} 2^{\frac{-j}{d-1}} \right) \\ &\sim \sum_{k=0}^{\infty} 2^{-k} |E_k| 2^{\frac{-k}{d-1}} = \sum_{k=0}^{\infty} 2^{-\frac{dk}{d-1}} |E_k| = 1. \end{aligned}$$

Thus, we complete the proof. \square

4. THE PROOF OF THE INEQUALITY (1.6) IN THEOREM 1.3

We shall provide the complete proof of the second part of Theorem 1.3 which can be restated as follows.

Theorem 4.1. *Let A_C be the restricted averaging operator associated with the cone $C \subset (\mathbb{F}_q^d, dx)$ defined as in (1.2). Suppose that σ denotes the normalized surface measure on the cone C . Then, if $d \geq 4$ is even, the estimate*

$$(4.1) \quad \|A_C f\|_{L^{\frac{d^2-2d+2}{d-2}}(C, \sigma)} \lesssim \|f\|_{L^{\frac{d^2-2d+2}{d^2-3d+2}}(\mathbb{F}_q^d, dx)}$$

holds for all function f on \mathbb{F}_q^d .

Proof. By duality, it suffices to prove that if $d \geq 4$ is even, then

$$\|A_C^* h\|_{L^{\frac{d^2-2d+2}{d}}(\mathbb{F}_q^d, dx)} \lesssim \|h\|_{L^{\frac{d^2-2d+2}{d^2-3d+4}}(C, \sigma)} \quad \text{for all } h : (C, \sigma) \rightarrow \mathbb{C},$$

where we recall from (2.1) that for $x \in (\mathbb{F}_q^d, dx)$

$$A_C^* h(x) = \frac{q^{2d}}{|C|^2} (h * C)(x).$$

Put $r = \frac{d^2-2d+2}{d}$ and $p = \frac{d^2-2d+2}{d^2-3d+4}$. Then our task is to show that the estimate

$$(4.2) \quad \left\| \frac{q^{2d}}{|C|^2} (h * C) \right\|_{L^r(\mathbb{F}_q^d, dx)} \lesssim \|h\|_{L^p(C, \sigma)}$$

holds for all $h : (C, \sigma) \rightarrow \mathbb{C}$. As usual, we may assume that h is a nonnegative real valued function supported on the cone C . By normalization of h , we also assume that

$$(4.3) \quad \sum_{x \in C} |h(x)|^p = 1.$$

Furthermore, we may assume that the function h can be written by a step function

$$(4.4) \quad h(x) = \sum_{j=0}^{\infty} 2^{-j} F_j(x) \quad \text{for } x \in C,$$

where F_j 's are pairwise disjoint subsets of C . From (4.3) and (4.4), we also assume

$$(4.5) \quad \sum_{k=0}^{\infty} 2^{-pk} |F_k| = 1.$$

Hence, it is natural to assume that for every $k = 0, 1, \dots$,

$$(4.6) \quad |F_k| \leq 2^{pk}.$$

With the above assumptions on h , our problem is reduced to showing that if $d \geq 4$ is even, then

$$\left\| \frac{q^{2d}}{|C|^2} (h * C) \right\|_{L^r(\mathbb{F}_q^d, dx)} \lesssim \|h\|_{L^p(C, \sigma)}.$$

Now recall from (2.8) and (2.9) that the characteristic function on the cone C is written by

$$C(x) = \widehat{M}(x) + \frac{|C|}{q^d} \quad \text{for } x \in (\mathbb{F}_q^d, dx),$$

where the function M on (\mathbb{F}_q^d, dm) is defined by $M(m) = C^\vee(m) - \frac{|C|}{q^d} \delta_0(m)$. Then, to complete the proof, it will be enough to show that if $d \geq 4$ is even, then we have

$$(4.7) \quad \left\| \frac{q^d}{|C|} (h * 1) \right\|_{L^r(\mathbb{F}_q^d, dx)} \lesssim \|h\|_{L^p(C, \sigma)}$$

and

$$(4.8) \quad \left\| \frac{q^{2d}}{|C|^2} (h * \widehat{M}) \right\|_{L^r(\mathbb{F}_q^d, dx)} \lesssim \|h\|_{L^p(C, \sigma)} := |C|^{-\frac{1}{p}} \left(\sum_{x \in C} |h(x)|^p \right)^{\frac{1}{p}},$$

where $r = \frac{d^2 - 2d + 2}{d}$, $p = \frac{d^2 - 2d + 2}{d^2 - 3d + 4}$, and the function h satisfies (4.3), (4.4), (4.5), (4.6). The estimate (4.7) simply follows by using Young's inequality for convolution functions. Indeed, it follows that

$$\begin{aligned} \left\| \frac{q^d}{|C|} (h * 1) \right\|_{L^r(\mathbb{F}_q^d, dx)} &\leq \frac{q^d}{|C|} \|h\|_{L^1(\mathbb{F}_q^d, dx)} \|1\|_{L^r(\mathbb{F}_q^d, dx)} \\ &= \frac{q^d}{|C|} \|h\|_{L^1(\mathbb{F}_q^d, dx)} = \|h\|_{L^1(C, \sigma)} \leq \|h\|_{L^p(C, \sigma)}, \end{aligned}$$

where the last inequality follows because dx is the normalized counting measure and $1 < p$. Thus, it remains to prove the estimate (4.8). Using (4.3) with the facts that $|S| \sim q^{d-1}$ for $d \geq 4$ and $p = \frac{d^2 - 2d + 2}{d^2 - 3d + 4}$, the estimate (4.8) can be rewritten by

$$q^{\frac{d^3 - 2d^2 + 3d}{d^2 - 2d + 2}} \|h * \widehat{M}\|_{L^r(\mathbb{F}_q^d, dx)} \lesssim 1.$$

Since $\|h * \widehat{M}\|_{L^r(\mathbb{F}_q^d, dx)}^2 = \|(h * \widehat{M})(h * \widehat{M})\|_{L^{\frac{r}{2}}(\mathbb{F}_q^d, dx)}$, the above estimate is equivalent to the estimate

$$q^{\frac{2d^3-4d^2+6d}{d^2-2d+2}} \|(h * \widehat{M})(h * \widehat{M})\|_{L^{\frac{r}{2}}(\mathbb{F}_q^d, dx)} \lesssim 1,$$

which we must prove. By (4.4) and Minkowski's inequality, the left hand side of the above inequality is dominated by

$$\begin{aligned} & q^{\frac{2d^3-4d^2+6d}{d^2-2d+2}} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} 2^{-k-j} \|(F_k * \widehat{M})(F_j * \widehat{M})\|_{L^{\frac{r}{2}}(\mathbb{F}_q^d, dx)} \\ & \sim q^{\frac{2d^3-4d^2+6d}{d^2-2d+2}} \sum_{k=0}^{\infty} \sum_{j=k}^{\infty} 2^{-k-j} \|(F_k * \widehat{M})(F_j * \widehat{M})\|_{L^{\frac{r}{2}}(\mathbb{F}_q^d, dx)}, \end{aligned}$$

where the last line is obtained by the symmetry of k, j . Thus, our final task to complete the proof is to show that if $d \geq 4$ is even, then we have

$$(4.9) \quad B := q^{\frac{2d^3-4d^2+6d}{d^2-2d+2}} \sum_{k=0}^{\infty} \sum_{j=k}^{\infty} 2^{-k-j} \|(F_k * \widehat{M})(F_j * \widehat{M})\|_{L^{\frac{r}{2}}(\mathbb{F}_q^d, dx)} \lesssim 1,$$

where $r = \frac{d^2-2d+2}{d}$ and we assume that (4.5) and (4.6) hold with $p = \frac{d^2-2d+2}{d^2-3d+4}$. In the following subsections, we shall prove the estimate (4.9) in the case when $d \geq 6$ and $d = 4$, respectively, and so the proof of Theorem 4.1 will be complete.

4.1. Proof of the estimate (4.9) for even dimensions $d \geq 6$. Assume that $d \geq 6$ is even. From (2.10) of Lemma 2.8 and (2.14) of Lemma 2.10, we see that

$$\begin{aligned} (4.10) \quad B & \leq q^{\frac{2d^3-4d^2+6d}{d^2-2d+2}} \sum_{k=0}^{\infty} \sum_{j=k}^{\infty} 2^{-k-j} \|F_k * \widehat{M}\|_{L^\infty(\mathbb{F}_q^d, dx)} \|F_j * \widehat{M}\|_{L^{\frac{r}{2}}(\mathbb{F}_q^d, dx)} \\ & \lesssim \sum_{k=0}^{\infty} \sum_{j=k}^{\infty} 2^{-k-j} |F_k| |F_j|^{\frac{d^2-4d+6}{d^2-2d+2}} \end{aligned}$$

Since $p = \frac{d^2-2d+2}{d^2-3d+4}$, it follows from (4.6) and (4.5) that

$$|F_j| \leq 2^{\frac{(d^2-2d+2)j}{d^2-3d+4}} \text{ for all } j = 0, 1, \dots \text{ and } \sum_{k=0}^{\infty} 2^{-\frac{(d^2-2d+2)k}{d^2-3d+4}} |F_k| = 1.$$

Using these facts, we conclude that

$$\begin{aligned} B & \lesssim \sum_{k=0}^{\infty} \sum_{j=k}^{\infty} 2^{-k-j} |F_k| 2^{\frac{(d^2-4d+6)j}{d^2-3d+4}} = \sum_{k=0}^{\infty} 2^{-k} |F_k| \left(\sum_{j=k}^{\infty} 2^{\frac{(-d+2)j}{d^2-3d+4}} \right) \\ & \sim \sum_{k=0}^{\infty} 2^{-k} |F_k| 2^{\frac{(-d+2)k}{d^2-3d+4}} = \sum_{k=0}^{\infty} 2^{-\frac{(d^2-2d+2)k}{d^2-3d+4}} |F_k| = 1. \end{aligned}$$

Thus the estimate (4.9) holds for even dimensions $d \geq 6$.

Remark 4.2. Recall that to deduce the inequality (4.10) we used the estimate (2.14) of Lemma 2.10 which was proved only for even dimension $d \geq 6$. However, if $d = 4$, we can not apply the estimate (2.14) of Lemma 2.10 and so we need to take a different approach to prove the estimate (4.9) for $d = 4$.

4.2. Proof of the estimate (4.9) for $d = 4$. We aim to show that

$$(4.11) \quad B := q^{\frac{44}{5}} \sum_{k=0}^{\infty} \sum_{j=k}^{\infty} 2^{-k-j} \|(F_k * \widehat{M})(F_j * \widehat{M})\|_{L^{\frac{5}{4}}(\mathbb{F}_q^4, dx)} \lesssim 1,$$

where the following conditions hold:

$$(4.12) \quad \sum_{k=0}^{\infty} 2^{-\frac{5k}{4}} |F_k| = 1$$

and

$$(4.13) \quad |F_k| \leq 2^{\frac{5k}{4}} \quad \text{for all } j = 0, 1, \dots$$

By Hölder's inequality, we have

$$B \leq q^{\frac{44}{5}} \sum_{k=0}^{\infty} \sum_{j=k}^{\infty} 2^{-k-j} \|(F_k * \widehat{M})\|_{L^{\frac{10}{3}}(\mathbb{F}_q^4, dx)} \|(F_j * \widehat{M})\|_{L^2(\mathbb{F}_q^4, dx)}.$$

Since $F_k \subset C \subset \mathbb{F}_q^4$ for $k = 0, 1, \dots$, and $|C| \sim q^3$, we have

$$\begin{aligned} B &\leq q^{\frac{44}{5}} \sum_{\substack{k=0 \\ :1 \leq |F_k| \leq q}}^{\infty} \sum_{j=k}^{\infty} 2^{-k-j} \|(F_k * \widehat{M})\|_{L^{\frac{10}{3}}(\mathbb{F}_q^4, dx)} \|(F_j * \widehat{M})\|_{L^2(\mathbb{F}_q^4, dx)} \\ &\quad + q^{\frac{44}{5}} \sum_{\substack{k=0 \\ :q \leq |F_k| \leq q^2}}^{\infty} \sum_{j=k}^{\infty} 2^{-k-j} \|(F_k * \widehat{M})\|_{L^{\frac{10}{3}}(\mathbb{F}_q^4, dx)} \|(F_j * \widehat{M})\|_{L^2(\mathbb{F}_q^4, dx)} \\ &\quad + q^{\frac{44}{5}} \sum_{\substack{k=0 \\ :q^2 \leq |F_k| \lesssim q^3}}^{\infty} \sum_{j=k}^{\infty} 2^{-k-j} \|(F_k * \widehat{M})\|_{L^{\frac{10}{3}}(\mathbb{F}_q^4, dx)} \|(F_j * \widehat{M})\|_{L^2(\mathbb{F}_q^4, dx)}. \end{aligned}$$

Using the upper bound of $\|F_k * \widehat{M}\|_{L^{\frac{10}{3}}(\mathbb{F}_q^4, dx)}$ in (2.15) of Lemma 2.10, we see that

$$\begin{aligned} B &\leq q^{\frac{9}{2}} \sum_{\substack{k=0 \\ :1 \leq |F_k| \leq q}}^{\infty} \sum_{j=k}^{\infty} 2^{-k-j} |F_k|^{\frac{7}{10}} \|(F_j * \widehat{M})\|_{L^2(\mathbb{F}_q^4, dx)} \\ &\quad + q^{\frac{21}{5}} \sum_{\substack{k=0 \\ :q \leq |F_k| \leq q^2}}^{\infty} \sum_{j=k}^{\infty} 2^{-k-j} |F_k| \|(F_j * \widehat{M})\|_{L^2(\mathbb{F}_q^4, dx)} \\ &\quad + q^{\frac{24}{5}} \sum_{\substack{k=0 \\ :q^2 \leq |F_k| \lesssim q^3}}^{\infty} \sum_{j=k}^{\infty} 2^{-k-j} |F_k|^{\frac{7}{10}} \|(F_j * \widehat{M})\|_{L^2(\mathbb{F}_q^4, dx)} \\ &:= B_1 + B_2 + B_3. \end{aligned}$$

To prove (4.11), it will be enough to show that

$$B_1 \lesssim 1, \quad B_2 \lesssim 1, \quad B_3 \lesssim 1.$$

Now recall from (2.16) and (2.18) that the estimates

$$(4.14) \quad \|F * \widehat{M}\|_{L^2(\mathbb{F}_q^4, dx)} \lesssim q^{-\frac{9}{2}} |F|^{\frac{3}{4}}$$

and

$$(4.15) \quad \|F * \widehat{M}\|_{L^2(\mathbb{F}_q^4, dx)} \lesssim \begin{cases} q^{-4}|F|^{\frac{1}{2}} & \text{if } q^2 \leq |F| \lesssim q^3 \\ q^{-5}|F| & \text{if } q \leq |F| \leq q^2 \\ q^{-\frac{9}{2}}|F|^{\frac{1}{2}} & \text{if } 1 \leq |F| \leq q \end{cases}$$

hold for all $F \subset (C, \sigma) \subset \mathbb{F}_q^4$. In order to estimate B_1 , we use the estimates (4.14), (4.13). Then it follows that

$$\begin{aligned} B_1 &\lesssim \sum_{\substack{k=0 \\ :1 \leq |F_k| \leq q}}^{\infty} \sum_{j=k}^{\infty} 2^{-k-j} |F_k|^{\frac{7}{10}} |F_j|^{\frac{3}{4}} \leq \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} 2^{-k-j} |F_k|^{\frac{7}{10}} |F_j|^{\frac{3}{4}} \\ &\leq \left(\sum_{k=0}^{\infty} 2^{-\frac{k}{8}} \right) \left(\sum_{j=0}^{\infty} 2^{-\frac{j}{16}} \right) \sim 1. \end{aligned}$$

Hence, $B_1 \lesssim 1$. Next, to estimate B_2 , we write

$$\begin{aligned} B_2 &= q^{\frac{21}{5}} \sum_{\substack{k=0 \\ :q \leq |F_k| \leq q^2}}^{\infty} \sum_{\substack{j=k \\ :1 \leq |F_j| < q^2}}^{\infty} 2^{-k-j} |F_k| \|(F_j * \widehat{M})\|_{L^2(\mathbb{F}_q^4, dx)} \\ &\quad + q^{\frac{21}{5}} \sum_{\substack{k=0 \\ :q \leq |F_k| \leq q^2}}^{\infty} \sum_{\substack{j=k \\ :q^2 \leq |F_j| \lesssim q^3}}^{\infty} 2^{-k-j} |F_k| \|(F_j * \widehat{M})\|_{L^2(\mathbb{F}_q^4, dx)} \\ &:= B_{2,1} + B_{2,2}. \end{aligned}$$

To estimate $B_{2,1}$, we use (4.14). Then we see that

$$B_{2,1} \lesssim q^{-\frac{3}{10}} \sum_{k=0}^{\infty} \sum_{\substack{j=k \\ :1 \leq |F_j| < q^2}}^{\infty} 2^{-k-j} |F_k| |F_j|^{\frac{3}{4}}.$$

Observe that if $|F_j| < q^2$, then $q^{-\frac{3}{10}} |F_j|^{\frac{3}{4}} < |F_j|^{\frac{3}{5}}$. From this observation, (4.13), and (4.12), it follows that

$$\begin{aligned} B_{2,1} &\lesssim \sum_{k=0}^{\infty} \sum_{j=k}^{\infty} 2^{-k-j} |F_k| |F_j|^{\frac{3}{5}} \leq \sum_{k=0}^{\infty} 2^{-k} |F_k| \left(\sum_{j=k}^{\infty} 2^{-\frac{j}{4}} \right) \\ &\sim \sum_{k=0}^{\infty} 2^{-k} |F_k| 2^{-\frac{k}{4}} = \sum_{k=0}^{\infty} 2^{-\frac{5k}{4}} |F_k| = 1. \end{aligned}$$

In order to estimate $B_{2,2}$, notice from (4.15) that

$$\|(F_j * \widehat{M})\|_{L^2(\mathbb{F}_q^4, dx)} \lesssim q^{-4} |F_j|^{\frac{1}{2}} \quad \text{if } q^2 \leq |F_j| \lesssim q^3.$$

Using this, we see that

$$B_{2,2} \lesssim q^{\frac{1}{5}} \sum_{k=0}^{\infty} \sum_{\substack{j=k \\ :q^2 \leq |F_j| \lesssim q^3}}^{\infty} 2^{-k-j} |F_k| |F_j|^{\frac{1}{2}}.$$

Since $q^{\frac{1}{5}}|F_j|^{\frac{1}{2}} \leq |F_j|^{\frac{3}{5}}$ for $|F_j| \geq q^2$, it follows that

$$B_{2,2} \lesssim \sum_{k=0}^{\infty} \sum_{j=k}^{\infty} 2^{-k-j} |F_k| |F_j|^{\frac{3}{5}}.$$

Applying (4.13) and (4.12), we have

$$B_{2,2} \lesssim \sum_{k=0}^{\infty} 2^{-k} |F_k| \left(\sum_{j=k}^{\infty} 2^{-\frac{j}{4}} \right) \sim \sum_{k=0}^{\infty} 2^{-k} |F_k| 2^{-\frac{k}{4}} = \sum_{k=0}^{\infty} 2^{-\frac{5k}{4}} |F_k| = 1.$$

Thus, we have proved that $B_2 \lesssim 1$. Finally, we shall prove $B_3 \lesssim 1$. To estimate B_3 , we split B_3 into two terms:

$$\begin{aligned} B_3 &= q^{\frac{24}{5}} \sum_{\substack{k=0 \\ :q^2 \leq |F_k| \lesssim q^3}}^{\infty} \sum_{\substack{j=k \\ :1 \leq |F_j| < q^2}}^{\infty} 2^{-k-j} |F_k|^{\frac{7}{10}} \|(F_j * \widehat{M})\|_{L^2(\mathbb{F}_q^4, dx)} \\ &\quad + q^{\frac{24}{5}} \sum_{\substack{k=0 \\ :q^2 \leq |F_k| \lesssim q^3}}^{\infty} \sum_{\substack{j=k \\ :q^2 \leq |F_j| \lesssim q^3}}^{\infty} 2^{-k-j} |F_k|^{\frac{7}{10}} \|(F_j * \widehat{M})\|_{L^2(\mathbb{F}_q^4, dx)} \\ &:= B_{3,1} + B_{3,2}. \end{aligned}$$

It follows from (4.14) that

$$B_{3,1} \lesssim q^{\frac{3}{10}} \sum_{\substack{k=0 \\ :q^2 \leq |F_k| \lesssim q^3}}^{\infty} \sum_{\substack{j=k \\ :1 \leq |F_j| < q^2}}^{\infty} 2^{-k-j} |F_k|^{\frac{7}{10}} |F_j|^{\frac{3}{4}}.$$

Since $q^{\frac{3}{10}}|F_k|^{\frac{7}{10}} \leq q^{-\frac{3}{10}}|F_k|$ for $q^2 \leq |F_k|$, we have

$$B_{3,1} \lesssim \sum_{\substack{k=0 \\ :q^2 \leq |F_k| \lesssim q^3}}^{\infty} \sum_{\substack{j=k \\ :1 \leq |F_j| < q^2}}^{\infty} 2^{-k-j} |F_k| q^{-\frac{3}{10}} |F_j|^{\frac{3}{4}}.$$

Using the fact that $q^{-\frac{3}{10}}|F_j|^{\frac{3}{4}} < |F_j|^{\frac{3}{5}}$ for $|F_j| < q^2$, we obtain that

$$B_{3,1} \lesssim \sum_{\substack{k=0 \\ :q^2 \leq |F_k| \lesssim q^3}}^{\infty} \sum_{\substack{j=k \\ :1 \leq |F_j| < q^2}}^{\infty} 2^{-k-j} |F_k| |F_j|^{\frac{3}{5}} \leq \sum_{k=0}^{\infty} \sum_{j=k}^{\infty} 2^{-k-j} |F_k| |F_j|^{\frac{3}{5}}.$$

By (4.13) and (4.12), we see that

$$\begin{aligned} B_{3,1} &\lesssim \sum_{k=0}^{\infty} 2^{-k} |F_k| \left(\sum_{j=k}^{\infty} 2^{-\frac{j}{4}} \right) \\ &\sim \sum_{k=0}^{\infty} 2^{-k} |F_k| 2^{-\frac{k}{4}} = \sum_{k=0}^{\infty} 2^{-\frac{5k}{4}} |F_k| = 1. \end{aligned}$$

In order to estimate $B_{3,2}$, we begin by recalling from (4.15) that

$$\|(F_j * \widehat{M})\|_{L^2(\mathbb{F}_q^4, dx)} \lesssim q^{-4} |F_j|^{\frac{1}{2}} \quad \text{if } q^2 \leq |F_j| \lesssim q^3.$$

From this estimate and the definition of $B_{3,2}$, it follows that

$$B_{3,2} \lesssim q^{\frac{4}{5}} \sum_{\substack{k=0 \\ :q^2 \leq |F_k| \lesssim q^3}}^{\infty} \sum_{\substack{j=k \\ :q^2 \leq |F_j| \lesssim q^3}}^{\infty} 2^{-k-j} |F_k|^{\frac{7}{10}} |F_j|^{\frac{1}{2}}.$$

Since $q^{\frac{4}{5}} |F_k|^{\frac{7}{10}} \leq q^{\frac{1}{5}} |F_k|$ for $|F_k| \geq q^2$, it follows that

$$B_{3,2} \lesssim \sum_{\substack{k=0 \\ :q^2 \leq |F_k| \lesssim q^3}}^{\infty} \sum_{\substack{j=k \\ :q^2 \leq |F_j| \lesssim q^3}}^{\infty} 2^{-k-j} |F_k| q^{\frac{1}{5}} |F_j|^{\frac{1}{2}}.$$

We apply a fact that $q^{\frac{1}{5}} |F_j|^{\frac{1}{2}} \leq |F_j|^{\frac{3}{5}}$ for $|F_j| \geq q^2$, and conclude by (4.13) and (4.12) that

$$\begin{aligned} B_{3,2} &\lesssim \sum_{\substack{k=0 \\ :q^2 \leq |F_k| \lesssim q^3}}^{\infty} \sum_{\substack{j=k \\ :q^2 \leq |F_j| \lesssim q^3}}^{\infty} 2^{-k-j} |F_k| |F_j|^{\frac{3}{5}} \leq \sum_{k=0}^{\infty} \sum_{j=k}^{\infty} 2^{-k-j} |F_k| |F_j|^{\frac{3}{5}} \\ &\leq \sum_{k=0}^{\infty} 2^{-k} |F_k| \left(\sum_{j=k}^{\infty} 2^{-j} \right) \sim \sum_{k=0}^{\infty} 2^{-k} |F_k| 2^{-\frac{k}{4}} = \sum_{k=0}^{\infty} 2^{-\frac{5k}{4}} |F_k| = 1. \end{aligned}$$

We have proved that $B_3 \lesssim 1$. Putting all estimates together, we complete the proof of the estimate (4.9) for $d = 4$. \square

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